Rigorous Renormalization Group

Lake Como school of Advanced Study

August 26, 2024

Introduction

- The school is devoted to an introduction to rigorous RG methods.

Introduction

- The school is devoted to an introduction to rigorous RG methods.
- The basic mathematical objects are Euclidean correlations (regularized functional integrals)

$$S(\mathbf{x}_1, .., \mathbf{x}_n) = \frac{\int P(d\Phi) e^{-V(\Phi)} \Phi_{\mathbf{x}_1} ... \Phi_{\mathbf{x}_n}}{\int P(d\Phi) e^{-V(\Phi)}}$$

where x are points in a suitable finite lattice Λ with step a and side L. Averaging over all the field configurations with weight $P(d\Phi) e^{-V(\Phi)}$.

Introduction

- The school is devoted to an introduction to rigorous RG methods.
- The basic mathematical objects are Euclidean correlations (regularized functional integrals)

$$S(\mathbf{x}_1, .., \mathbf{x}_n) = \frac{\int P(d\Phi) e^{-V(\Phi)} \Phi_{\mathbf{x}_1} ... \Phi_{\mathbf{x}_n}}{\int P(d\Phi) e^{-V(\Phi)}}$$

where x are points in a suitable finite lattice Λ with step a and side L. Averaging over all the field configurations with weight $P(d\Phi)e^{-V(\Phi)}$.
- One has also to consider two cases which can be called bosonic or fermionìc.

Functional integrals

- In the bosonic case ($\Phi = \phi$) $\phi_x \in \mathbb{R}$, $P(d\phi)$ is a Gaussian measure ($\prod_x d\phi_x e^{-1/2(\phi,A\phi)}$) and $V(\phi)$ is a sum over monomials in ϕ; in the ϕ^4 model $V = \sum_x \frac{\lambda}{4!}\phi_x^4$ (λ is the coupling). Finite dimensional but $O((L/a)^d)$ variables (in the one dimensional case $\sim \int_{-\infty}^{\infty} dx e^{-x^2-\lambda x^4}$)

Functional integrals

- In the bosonic case ($\Phi = \phi$) $\phi_x \in \mathbb{R}$, $P(d\phi)$ is a Gaussian measure ($\prod_x d\phi_x e^{-1/2(\phi, A\phi))}$) and $V(\phi)$ is a sum over monomials in ϕ; in the ϕ^4 model $V = \sum_x \frac{\lambda}{4!}\phi_x^4$ (λ is the coupling). Finite dimensional but $O((L/a)^d)$ variables (in the one dimensional case $\sim \int_{-\infty}^{\infty} dx e^{-x^2-\lambda x^4}$)
- In the fermionic case ($\Phi = \psi$) then $\psi_x, \bar{\psi}_x$ are Grassmann variables, (anticommutative $\{\psi_x, \psi_y\} = 0$) $P(d\psi)$ is a Gaussian Grassmann integration and V is a sum over monomials in the Grassmann variables. V quartic, Fermi model (*Tentativo*). Finite dimensional but $O((L/a)^d)$ variables. $\int d\psi \psi = 1$ and zero otherwise. $\int d\psi_4 d\psi_3 d\psi_2 d\psi_1 e^{\lambda\psi_1\psi_2\psi_3\psi_4} = \lambda$. [See Gallone lecture]

Functional integrals

- In the bosonic case ($\Phi = \phi$) $\phi_x \in \mathbb{R}$, $P(d\phi)$ is a Gaussian measure ($\prod_x d\phi_x e^{-1/2(\phi,A\phi)}$) and $V(\phi)$ is a sum over monomials in ϕ; in the ϕ^4 model $V = \sum_x \frac{\lambda}{4!}\phi_x^4$ (λ is the coupling). Finite dimensional but $O((L/a)^d)$ variables (in the one dimensional case $\sim \int_{-\infty}^{\infty} dx e^{-x^2-\lambda x^4}$)
- In the fermionic case ($\Phi = \psi$) then $\psi_x, \bar\psi_x$ are Grassmann variables, (anticommutative $\{\psi_x, \psi_y\} = 0$) $P(d\psi)$ is a Gaussian Grassmann integration and V is a sum over monomials in the Grassmann variables. V quartic, Fermi model (*Tentativo*). Finite dimensional but $O((L/a)^d)$ variables. $\int d\psi\psi = 1$ and zero otherwise. $\int d\psi_4 d\psi_3 d\psi_2 d\psi_1 e^{\lambda\psi_1\psi_2\psi_3\psi_4} = \lambda$. [See Gallone lecture]
- The problem is to compute the $|\Lambda| \to \infty$ limit (infrared problem) or the $a \to 0$ limit (ultraviolet limit) ; or both

Functional integrals

- In the bosonic case ($\Phi=\phi$) $\phi_x \in \mathbb{R}$, $P(d\phi)$ is a Gaussian measure ($\prod_x d\phi_x e^{-1/2(\phi, A\phi)}$) and $V(\phi)$ is a sum over monomials in ϕ; in the ϕ^4 model $V=\sum_x \frac{\lambda}{4!}\phi_x^4$ (λ is the coupling). Finite dimensional but $O((L/a)^d)$ variables (in the one dimensional case $\sim \int_{-\infty}^{\infty} dx e^{-x^2-\lambda x^4}$)
- In the fermionic case ($\Phi=\psi$) then $\psi_x, \bar\psi_x$ are Grassmann variables, (anticommutative $\{\psi_x, \psi_y\}=0$) $P(d\psi)$ is a Gaussian Grassmann integration and V is a sum over monomials in the Grassmann variables. V quartic, Fermi model (*Tentativo*). Finite dimensional but $O((L/a)^d)$ variables. $\int d\psi\psi=1$ and zero otherwise. $\int d\psi_4 d\psi_3 d\psi_2 d\psi_1 e^{\lambda\psi_1\psi_2\psi_3\psi_4}=\lambda$. [See Gallone lecture]
- The problem is to compute the $|\Lambda|\to\infty$ limit (infrared problem) or the $a\to 0$ limit (ultraviolet limit) ; or both
- We will focus on the Grassmann integrals.

Applications

- The above objects are central in modern physics in all kinds of phenomena in which the interaction between particles produces modifications with respect to the non intercting behavior. In particular

Applications

- The above objects are central in modern physics in all kinds of phenomena in which the interaction between particles produces modifications with respect to the non intercting behavior. In particular
- In condensed matter conduction electrons in metals are a gas of interacting fermions; interaction can produce dramatic effects like superconductvity, luttinger liquid behavior etc. The equilibrium correlations and transport coefficients can be written as grassmann integrals (Matsubara) Λ is the space- (imaginary) time, side β and L. One is interested in $L \to \infty$ (thermodynamic limit) and zero temperature $\beta \to \infty$) (see Porta lecture) (x_1 discrete and x_0 continuous).

Applications

- Also classical statistical mechanics models, like the 2D Ising model and its perturbations, or dimers, vertex models and so on can be written in terms of Grassmann integrals. There Λ is the physical lattice, and the deviation from the critical temperature is the mass of fermions (see Gallone lecture).

Applications

- Also classical statistical mechanics models, like the 2D Ising model and its perturbations, or dimers, vertex models and so on can be written in terms of Grassmann integrals. There Λ is the physical lattice, and the deviation from the critical temperature is the mass of fermions (see Gallone lecture).
- Models in Euclidean QFT, like QED, Standard Model etc regularized on a lattice are expressed functional integrals. Feynman integral $e^{\frac{-iS}{\hbar}}$ with Wick rotation. Purely fermionic ones are the fermi theory of weak interactions, the Thirring model and several others (fermions are particles and bosons the fields).

Applications

- Also classical statistical mechanics models, like the 2D Ising model and its perturbations, or dimers, vertex models and so on can be written in terms of Grassmann integrals. There Λ is the physical lattice, and the deviation from the critical temperature is the mass of fermions (see Gallone lecture).
- Models in Euclidean QFT, like QED, Standard Model etc regularized on a lattice are expressed functional integrals. Feynman integral $e^{\frac{-iS}{\hbar}}$ with Wick rotation. Purely fermionic ones are the fermi theory of weak interactions, the Thirring model and several others (fermions are particles and bosons the fields).
- The construction of a QFT consist in proving that one cam take the limit $L \to \infty$, $a \to 0$ and verify Osterwalder-Schrader and then Wightmann axioms. If a fxed one construct an effective QFT.

Applications

- Also classical statistical mechanics models, like the 2D Ising model and its perturbations, or dimers, vertex models and so on can be written in terms of Grassmann integrals. There Λ is the physical lattice, and the deviation from the critical temperature is the mass of fermions (see Gallone lecture).
- Models in Euclidean QFT, like QED, Standard Model etc regularized on a lattice are expressed functional integrals. Feynman integral $e^{\frac{-iS}{\hbar}}$ with Wick rotation. Purely fermionic ones are the fermi theory of weak interactions, the Thirring model and several others (fermions are particles and bosons the fields).
- The construction of a QFT consist in proving that one cam take the limit $L \to \infty$, $a \to 0$ and verify Osterwalder-Schrader and then Wightmann axioms. If a fxed one construct an effective QFT.
- The theory of such functional ntegrals has been originated by some many different physical domanis and the language to describe the property feels such variety.

Dressed and bare quantities

- One can try to adopt a perturbative method expanding e^V; this however typically fails in the thermodynamic limit or at criticality. In QFT (in the continuum) there is also a basic problem connected to the fact that the integrals are diverging at short distances (ultraviolet problem)

Dressed and bare quantities

- One can try to adopt a perturbative method expanding e^V; this however typically fails in the thermodynamic limit or at criticality. In QFT (in the continuum) there is also a basic problem connected to the fact that the integrals are diverging at short distances (ultraviolet problem)
- The interaction can produce a radically different physical behavior (how we can recostruct perturbatively?) For instance in condensed matter you can have Luttinger behaviour (the occupation number is not discpntinuous but continuous) or superconductivity

Dressed and bare quantities

- One can try to adopt a perturbative method expanding e^V; this however typically fails in the thermodynamic limit or at criticality. In QFT (in the continuum) there is also a basic problem connected to the fact that the integrals are diverging at short distances (ultraviolet problem)
- The interaction can produce a radically different physical behavior (how we can recostruct perturbatively?) For instance in condensed matter you can have Luttinger behaviour (the occupation number is not discpntinuous but continuous) or superconductivity
- Even in the more normal cases (like Fermi liquid) the parameters are always modified or dressed (the Fermi momentum the mass in QFT, the critical temperature in stat phys), so expansion in λ does not work. In QFT the bare quantities diverge with the uv cut-off.

Dressed and bare quantities

- One can try to adopt a perturbative method expanding e^V; this however typically fails in the thermodynamic limit or at criticality. In QFT (in the continuum) there is also a basic problem connected to the fact that the integrals are diverging at short distances (ultraviolet problem)
- The interaction can produce a radically different physical behavior (how we can recostruct perturbatively?) For instance in condensed matter you can have Luttinger behaviour (the occupation number is not discpntinuous but continuous) or superconductivity
- Even in the more normal cases (like Fermi liquid) the parameters are always modified or dressed (the Fermi momentum the mass in QFT, the critical temperature in stat phys), so expansion in λ does not work. In QFT the bare quantities diverge with the uv cut-off.
- In pert theory $\frac{1}{k^2+(m^2+\lambda^2)}$ appears as $\frac{1}{k^2+m^2}\sum_{n=0}^{\infty}(-1)^n(\frac{\lambda^2}{k^2+m^2})^n$

Universality

- Somewhat paradoxically, some quantity is instead completely independent from interaction and microscopic detail

Universality

- Somewhat paradoxically, some quantity is instead completely independent from interaction and microscopic detail
- This is known from experiments. Experiments show that the exponents are the exactly same in a wide class of system and often coincide with Ising; for instance in $d = 2$ the index β at the ferromagnetic transition is 0.119(8) in $Rb_2 CoF_4$, 0.123(8) in $K_2 CoF_4$, 0.135(3) in $Ba_2 FeF_6$. ($\beta = 1/8$ in Ising); in $d = 3$ $\nu = 0.60.7$ and best numerical results $\nu = 0,6299...$

Universality

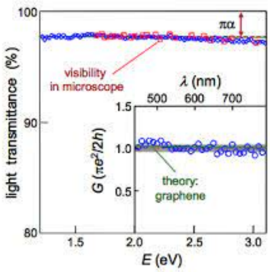
- Somewhat paradoxically, some quantity is instead completely independent from interaction and microscopic detail
- This is known from experiments. Experiments show that the exponents are the exactly same in a wide class of system and often coincide with Ising; for instance in $d = 2$ the index β at the ferromagnetic transition is 0.119(8) in $Rb_2 CoF_4$, 0.123(8) in $K_2 CoF_4$, 0.135(3) in $Ba_2 FeF_6$. ($\beta = 1/8$ in Ising); in $d = 3$ $\nu = 0.60.7$ and best numerical results $\nu = 0, 6299...$
- In condensed matter the Hall conductivity

$$\sigma = ne^2/h$$

with n integer,

Universality

- The optical conductivity of graphene Experiments (Geim Nosovelov..(2008) show no interaction corrections



The Fermi velocity is drastically renormalized instead

Universality

- The exponents can be modified and be non universal but can veify relations allowing th express them exactly in terms of a single one .marginal universality.

Universality

- The exponents can be modified and be non universal but can veify relations allowing th express them exactly in terms of a single one .marginal universality.
- In Luttinger liquids (Haldane 1980)

$$D = v_s K/\pi \qquad \eta = (K+1/K^{-}2)/2 \quad \nu = 2/(1-1/K) \quad X_+ = 1/X_- = K, \kappa = K/(\pi v_s)$$

Universality

- The exponents can be modified and be non universal but can veify relations allowing th express them exactly in terms of a single one .marginal universality.
- In Luttinger liquids (Haldane 1980)

$$D = v_s K/\pi \qquad \eta = (K+1/K^{-}2)/2 \quad \nu = 2/(1-1/K) \quad X_+ = 1/X_- = K, \kappa = K/(\pi v_s)$$

- Verified in solvable models like Luttinger or XXZ; even the slightest modification destroy solvabililty

Renormalization

- First analysis of functional integrals in QED lead to ultraviolet divergences; integrals are infinite

Renormalization

- First analysis of functional integrals in QED lead to ultraviolet divergences; integrals are infinite
- renormalization, $\sim$ 1948 (Bethe, Schwinger, Feynman, Tomonaga, Dyson,..). Observable P as series in the (bare) charge e_b; $P(e_b) = P_0 + P_1 e_b + P_2 e_b^2 + ...$ with $P_i = \infty$ infinite!

Renormalization

- First analysis of functional integrals in QED lead to ultraviolet divergences; integrals are infinite
- renormalization, ~ 1948 (Bethe, Schwinger, Feynman, Tomonaga, Dyson,..). Observable P as series in the (bare) charge e_b; $P(e_b) = P_0 + P_1 e_b + P_2 e_b^2 + ...$ with $P_i = \infty$ infinite!
- One introduces a regularization Λ, $P(e_b, \Lambda) = P_0(\Lambda) + P_1(\Lambda) e_b + ...$; Among the P one can identify the dressed charge, which is the measured one $e_d(e_b, \Lambda)$; formally inverting $e_b = e_b(\Lambda, e_d)$ we get $P(e_b(\Lambda, e_d), \Lambda) = \tilde{P}_0(\Lambda) + \tilde{P}_1(\Lambda) e_d + \tilde{P}_2(\Lambda) e_d^2 + ...$

Renormalization

- First analysis of functional integrals in QED lead to ultraviolet divergences; integrals are infinite
- renormalization, $\sim$ 1948 (Bethe, Schwinger, Feynman, Tomonaga, Dyson,..). Observable P as series in the (bare) charge e_b; $P(e_b) = P_0 + P_1 e_b + P_2 e_b^2 + ...$ with $P_i = \infty$ infinite!
- One introduces a regularization Λ, $P(e_b, \Lambda) = P_0(\Lambda) + P_1(\Lambda) e_b + ...$; Among the P one can identify the dressed charge, which is the measured one $e_d(e_b, \Lambda)$; formally inverting $e_b = e_b(\Lambda, e_d)$ we get $P(e_b(\Lambda, e_d), \Lambda) = \tilde{P}_0(\Lambda) + \tilde{P}_1(\Lambda) e_d + \tilde{P}_2(\Lambda) e_d^2 + ...$
- In QED or in the SM then $\lim_{\Lambda\to\infty} \tilde{P}_i(\Lambda) < \infty$ finite!; of course one needs to face convergence (typically not convergent, triviality and all that)

Renormalization

- First analysis of functional integrals in QED lead to ultraviolet divergences; integrals are infinite
- renormalization, $\sim$ 1948 (Bethe, Schwinger, Feynman, Tomonaga, Dyson,..). Observable P as series in the (bare) charge e_b; $P(e_b) = P_0 + P_1 e_b + P_2 e_b^2 + ...$ with $P_i = \infty$ infinite!
- One introduces a regularization Λ, $P(e_b, \Lambda) = P_0(\Lambda) + P_1(\Lambda) e_b + ...$; Among the P one can identify the dressed charge, which is the measured one $e_d(e_b, \Lambda)$; formally inverting $e_b = e_b(\Lambda, e_d)$ we get $P(e_b(\Lambda, e_d), \Lambda) = \tilde{P}_0(\Lambda) + \tilde{P}_1(\Lambda) e_d + \tilde{P}_2(\Lambda) e_d^2 + ...$
- In QED or in the SM then $\lim_{\Lambda\to\infty} \tilde{P}_i(\Lambda) < \infty$ finite!; of course one needs to face convergence (typically not convergent, triviality and all that)
- The Renormalization Group is the modern way of implementing renormalziation.

Renormalization Group

- Renormalization Group was introduced by Wilson (1975) was introduced. Crucial notion of scaling, dimension and relevant or irrelevant terms. It was a revolution making quantitative the concept of emergence of a certain behavior at a certain length scale to others.

Renormalization Group

- Renormalization Group was introduced by Wilson (1975) was introduced. Crucial notion of scaling, dimension and relevant or irrelevant terms. It was a revolution making quantitative the concept of emergence of a certain behavior at a certain length scale to others.
- Essentially the propagator or covariance $\hat{g}_\Phi(k)$ is written as sum of propagators $\sum_{h=-\infty}^{N} \hat{g}_\Phi^h(k)$ each non vanishing for $c\gamma^{h-1} \leq |k| \leq c\gamma^{h+1}$ for $h \leq N-1$, $\gamma > 1$

Renormalization Group

- Renormalization Group was introduced by Wilson (1975) was introduced. Crucial notion of scaling, dimension and relevant or irrelevant terms. It was a revolution making quantitative the concept of emergence of a certain behavior at a certain length scale to others.
- Essentially the propagator or covariance $\hat{g}_\Phi(k)$ is written as sum of propagators $\sum_{h=-\infty}^{N} \hat{g}_\Phi^h(k)$ each non vanishing for $c\gamma^{h-1} \leq |k| \leq c\gamma^{h+1}$ for $h \leq N-1$, $\gamma > 1$
- By addition property of Gaussian integrals

$$\int P(d\Phi) e^{-V(\Phi)} = \int P(d\Phi^{<N}) \int P(d\Phi^N) e^{-V(\Phi^{<N}+\Phi^N))} = \int P(d\Phi^{<N}) e^{-V^N(\Phi^{<N})}$$

where $P(d\Phi^{<N})$ has propagator $\sum_{h=-\infty}^{N-1} \hat{g}_\Phi^h(k)$, $P(d\Phi^N)$ has propagator $\hat{g}_\Phi^N(k)$ and V^N is sum of monomials of any order in $\phi^{<N}$ integrated over certain kernels.

Renormalization Group

- The procedure can be iterated obtaining V^h. During such procedure some of the terms in V^h tend to increase and others to decrease.

Renormalization Group

- The procedure can be iterated obtaining V^h. During such procedure some of the terms in V^h tend to increase and others to decrease.
- Supposes that the propagator is $\hat{g}(\mathbf{k}) = \chi/|\mathbf{k}|^\alpha$; then

$$g_{\Phi}^{\leq h}(x) \sim \gamma^{(h-N)(d-\alpha)} g_{\Phi}^{\leq N}(\gamma^{(h-N)}x)$$

Renormalization Group

- The procedure can be iterated obtaining V^h. During such procedure some of the terms in V^h tend to increase and others to decrease.
- Supposes that the propagator is $\hat{g}(\mathbf{k}) = \chi/|\mathbf{k}|^\alpha$; then

$$g_{\Phi}^{\leq h}(x) \sim \gamma^{(h-N)(d-\alpha)} g_{\Phi}^{\leq N}(\gamma^{(h-N)} x)$$

- The field with cut-off at scale h $\Phi^{\leq h}(x)$ is essentially distributed as $\gamma^{(h-N)(d-\alpha)/2}$ $\Phi^{\leq N}(\gamma^{(h-N)} x)$.

Renormalization Group

- The procedure can be iterated obtaining V^h. During such procedure some of the terms in V^h tend to increase and others to decrease.
- Supposes that the propagator is $\hat{g}(\mathbf{k}) = \chi/|\mathbf{k}|^\alpha$; then

$$g_{\Phi}^{\leq h}(x) \sim \gamma^{(h-N)(d-\alpha)} g_{\Phi}^{\leq N}(\gamma^{(h-N)}x)$$

- The field with cut-off at scale h $\Phi^{\leq h}(x)$ is essentially distributed as $\gamma^{(h-N)(d-\alpha)/2}$ $\Phi^{\leq N}(\gamma^{(h-N)}x)$.
- The monomials appearing in V^h of the form $\int dx O_n^{\leq h}$, a local monomial with n fields, behaves essentially as $O_n^{\leq N}$ with a prefactor

$$\gamma^{-(h-N)D_n} \int dx O_n^{\leq N} \quad D_n = d - \frac{(d-\alpha)}{2} n$$

Renormalization Group

- Therefore after the integration of the field at scales $N, N-1, .., h$ one obtains an expression similar to the initial one but where each monomial in the potential is multiplied by a factor $\gamma^{-(h-N)D_n}$.

Renormalization Group

- Therefore after the integration of the field at scales $N, N-1, .., h$ one obtains an expression similar to the initial one but where each monomial in the potential is multiplied by a factor $\gamma^{-(h-N)D_n}$.
- The terms with $D_n > 0$ are increased integrating the fields from scale N to scale h; they are called relevant (superrenormalizable). The terms $D_n < 0$ are decreased and are called irrelevant (non renormalizable). The terms with $D_n = 0$ are unchanged by this scaling argument and are called marginal (renormalizable).

Renormalization Group

- Therefore after the integration of the field at scales $N, N-1, .., h$ one obtains an expression similar to the initial one but where each monomial in the potential is multiplied by a factor $\gamma^{-(h-N)D_n}$.
- The terms with $D_n > 0$ are increased integrating the fields from scale N to scale h; they are called relevant (superrenormalizable). The terms $D_n < 0$ are decreased and are called irrelevant (non renormalizable). The terms with $D_n = 0$ are unchanged by this scaling argument and are called marginal (renormalizable).
- When the interaction is irrelevant than it decreases iterating the RG; this essentially says that the large distance behavior (infared behavior) is unchanged; if relevant it increases and the behavior can be drastically changed. There are however cases in which the increasing is prevented by cancellations.

Renormalization Group

- Therefore after the integration of the field at scales $N, N-1, .., h$ one obtains an expression similar to the initial one but where each monomial in the potential is multiplied by a factor $\gamma^{-(h-N)D_n}$.
- The terms with $D_n > 0$ are increased integrating the fields from scale N to scale h; they are called relevant (superrenormalizable). The terms $D_n < 0$ are decreased and are called irrelevant (non renormalizable). The terms with $D_n = 0$ are unchanged by this scaling argument and are called marginal (renormalizable).
- When the interaction is irrelevant than it decreases iterating the RG; this essentially says that the large distance behavior (infared behavior) is unchanged; if relevant it increases and the behavior can be drastically changed. There are however cases in which the increasing is prevented by cancellations.
- In condensed matter typically $N = 0$ fixed. In QFT one wants to take $N \to \infty$; when irrelevant one needs a bare value larger and larger to be $O(1)$ at scale 0 (so outside perturbative accessible regime); when relevant the opposite

Wilsonian Renormalization Group

- When the interaction is marginal (renormalizable), the behavior is given by an equation called Beta function which has typically the form $\lambda_{h-1} = \lambda_h + a_h \lambda_h^2 + b_h \lambda_h^3 + ...$, the rrcc λ_h the effective coupling.

Wilsonian Renormalization Group

- When the interaction is marginal (renormalizable), the behavior is given by an equation called Beta function which has typically the form $\lambda_{h-1} = \lambda_h + a_h \lambda_h^2 + b_h \lambda_h^3 + ...$, the rrcc λ_h the effective coupling.
- If (roughly speaking) $a_h < 0$ then λ_h decrease; the i.r. behavior is the same as of the free theory (in uv increases) . If $a_h > 0$ it increases (maybe non trivial fixed point).; in the uv trivial fixed point. Marginally relevant or irrelevant can be decided on the basis on a second order analysis.

Wilsonian Renormalization Group

- When the interaction is marginal (renormalizable), the behavior is given by an equation called Beta function which has typically the form $\lambda_{h-1} = \lambda_h + a_h \lambda_h^2 + b_h \lambda_h^3 + \ldots$, the rrcc λ_h the effective coupling.
- If (roughly speaking) $a_h < 0$ then λ_h decrease; the i.r. behavior is the same as of the free theory (in uv increases) . If $a_h > 0$ it increases (maybe non trivial fixed point).; in the uv trivial fixed point. Marginally relevant or irrelevant can be decided on the basis on a second order analysis.
- Most rigorous results regard the case of irrelevant or marginally irrelevant for the ir (or relevant or marginally relevant for the uv). In the relevant or marginally relevant one expects "a non trivial fixed point".

Wilsonian Renormalization Group

- When the interaction is marginal (renormalizable), the behavior is given by an equation called Beta function which has typically the form $\lambda_{h-1} = \lambda_h + a_h \lambda_h^2 + b_h \lambda_h^3 + ...$, the rrcc λ_h the effective coupling.
- If (roughly speaking) $a_h < 0$ then λ_h decrease; the i.r. behavior is the same as of the free theory (in uv increases) . If $a_h > 0$ it increases (maybe non trivial fixed point).; in the uv trivial fixed point. Marginally relevant or irrelevant can be decided on the basis on a second order analysis.
- Most rigorous results regard the case of irrelevant or marginally irrelevant for the ir (or relevant or marginally relevant for the uv). In the relevant or marginally relevant one expects "a non trivial fixed point".
- In the marginally marginal case all coefficients are vanishing. This cannot be proved by perturbative computation and correspond to a line of fixed points. Lines of fixed points, different physical behavior.

Rigorous fermionic RG

- In most physical applications a)the irrelevant terms are neglected; b)higher orders are neglected and series are not convergent. In QFT (in the continuum) there are a basic problem connected to the fact that the integrals are diverging at short distances (ultraviolet problem)

Rigorous fermionic RG

- In most physical applications a)the irrelevant terms are neglected; b)higher orders are neglected and series are not convergent. In QFT (in the continuum) there are a basic problem connected to the fact that the integrals are diverging at short distances (ultraviolet problem)
- In condensed matter or statistical physics (part of) space time is discrete but there are problems connected to large distance behavior.

Rigorous fermionic RG

- In most physical applications a)the irrelevant terms are neglected; b)higher orders are neglected and series are not convergent. In QFT (in the continuum) there are a basic problem connected to the fact that the integrals are diverging at short distances (ultraviolet problem)
- In condensed matter or statistical physics (part of) space time is discrete but there are problems connected to large distance behavior.
- Physical information are extracted by certain ”tricks” like ”forbidden resummations” (self energy, like $1-1+1-1... = 1/2$) or truncations.

Rigorous fermionic RG

- In most physical applications a)the irrelevant terms are neglected; b)higher orders are neglected and series are not convergent. In QFT (in the continuum) there are a basic problem connected to the fact that the integrals are diverging at short distances (ultraviolet problem)
- In condensed matter or statistical physics (part of) space time is discrete but there are problems connected to large distance behavior.
- Physical information are extracted by certain "tricks" like "forbidden resummations" (self energy, like $1-1+1-1... = 1/2$) or truncations.
- The situation is remarkably similar to the debate on Newton mathematics in *XVIII* (eg Berkeley Maclaurin,..) on diverging series, fluxions, division by zero....

Rigorous fermionic RG

- In most physical applications a)the irrelevant terms are neglected; b)higher orders are neglected and series are not convergent. In QFT (in the continuum) there are a basic problem connected to the fact that the integrals are diverging at short distances (ultraviolet problem)
- In condensed matter or statistical physics (part of) space time is discrete but there are problems connected to large distance behavior.
- Physical information are extracted by certain "tricks" like "forbidden resummations" (self energy, like $1 - 1 + 1 - 1 ... = 1/2$) or truncations.
- The situation is remarkably similar to the debate on Newton mathematics in *XVIII* (eg Berkeley Maclaurin,..) on diverging series, fluxions, division by zero....
- Berkeley;"By a double error you arrived if not to science to the truth"compare with Feynman: "I suspect that renormalization is not mathematically legitimate."

Rigorous fermionic RG

- In most physical applications a)the irrelevant terms are neglected; b)higher orders are neglected and series are not convergent. In QFT (in the continuum) there are a basic problem connected to the fact that the integrals are diverging at short distances (ultraviolet problem)
- In condensed matter or statistical physics (part of) space time is discrete but there are problems connected to large distance behavior.
- Physical information are extracted by certain "tricks" like "forbidden resummations" (self energy, like $1-1+1-1... = 1/2$) or truncations.
- The situation is remarkably similar to the debate on Newton mathematics in $XVIII$ (eg Berkeley Maclaurin,..) on diverging series, fluxions, division by zero....
- Berkeley;"By a double error you arrived if not to science to the truth"compare with Feynman: "I suspect that renormalization is not mathematically legitimate."
- Starting from the 80's rigorous renormalization was introduced.

Rigorous fermionic RG

- In a rigorous context by "perturbative results" one means that one gets series finite removing the cut-off, in particular obtaining $n!$ bounds. Non-perturbative that the functional integrals are controlled removing some of the cut-off.

Rigorous fermionic RG

- In a rigorous context by "perturbative results" one means that one gets series finite removing the cut-off, in particular obtaining $n!$ bounds. Non-perturbative that the functional integrals are controlled removing some of the cut-off.
- The series for bosons around zero cannot be convergent in contrast to to the one for fermions, $\sim \int_{-\infty}^{\infty} dx e^{-x^2-\lambda x^4}$ versus $\int d\psi_4 d\psi_3 d\psi_2 d\psi_1 e^{\lambda\psi_1\psi_2\psi_3\psi_4} = \lambda$. The $n!$ are there for bosons but not for fermions.

Rigorous fermionic RG

- In a rigorous context by "perturbative results" one means that one gets series finite removing the cut-off, in particular obtaining $n!$ bounds. Non-perturbative that the functional integrals are controlled removing some of the cut-off.
- The series for bosons around zero cannot be convergent in contrast to to the one for fermions, $\sim \int_{-\infty}^{\infty} dx e^{-x^2-\lambda x^4}$ versus $\int d\psi_4 d\psi_3 d\psi_2 d\psi_1 e^{\lambda\psi_1\psi_2\psi_3\psi_4} = \lambda$. The $n!$ are there for bosons but not for fermions.
- Fermion series are convergent (good); however bosons have also advantages as can use saddle point arguments (one can decide to write fermions as boson by Hubbard Strataniwich or integrate the bosons considering only fermions).

Rigorous fermionic RG: QFT

- Gallavotti introduced a rigorous version of RG and the tree expansion (see review RMP 1986) obtaining $n!$ bounds.

Rigorous fermionic RG: QFT

- Gallavotti introduced a rigorous version of RG and the tree expansion (see review RMP 1986) obtaining $n!$ bounds.
- Gawedzki, Kupiainen (CMP 1986) constructed non-perturbatively the massive GN_2 model $D = 2 - n/2$ in the ultraviolet, which is renormalizable (among first examples) and marginally relevant (trivial uv fixed point); Lesniewski (CMP 1987) constructed the Yukawa_2 model which is superrenormalizable using Brydges formula (Brydges. Les Houches, 1980 and Gram bounds (Caianiello 1956) .

Rigorous fermionic RG: QFT

- Gallavotti introduced a rigorous version of RG and the tree expansion (see review RMP 1986) obtaining $n!$ bounds.
- Gawedzki, Kupiainen (CMP 1986) constructed non-perturbatively the massive GN_2 model $D = 2 - n/2$ in the ultraviolet, which is renormalizable (among first examples) and marginally relevant (trivial uv fixed point); Lesniewski (CMP 1987) constructed the $Yukawa_2$ model which is superrenormalizable using Brydges formula (Brydges. Les Houches, 1980 and Gram bounds (Caianiello 1956) .
- The uv GN_2 was also constructed by Feldman, Magnen, Rivasseau, Sénéor (CMP 1986) and the massless infrared in the large N limit with mass generation by Kopper, Magnen, Rivasseau (1995) (intermediate regime missing!).

Rigorous fermionic RG: QFT

- Gallavotti introduced a rigorous version of RG and the tree expansion (see review RMP 1986) obtaining $n!$ bounds.
- Gawedzki, Kupiainen (CMP 1986) constructed non-perturbatively the massive GN_2 model $D = 2 - n/2$ in the ultraviolet, which is renormalizable (among first examples) and marginally relevant (trivial uv fixed point); Lesniewski (CMP 1987) constructed the Yukawa_2 model which is superrenormalizable using Brydges formula (Brydges. Les Houches, 1980 and Gram bounds (Caianiello 1956) .
- The uv GN_2 was also constructed by Feldman, Magnen, Rivasseau, Sénéor (CMP 1986) and the massless infrared in the large N limit with mass generation by Kopper, Magnen, Rivasseau (1995) (intermediate regime missing!).
- Not so many other examples of QFT construction and OS axioms (Thirring model $d = 2$, Benfatto Falco Mastropietro 2006); in $d = 4$ effective QFT point of view (a small but finite), triviality of ϕ^4 (Aizenman Duminil-Copin Ann of Math 2020) makes SM probably effective by Higgs; possible QFT for strong sector.

Rigorous fermionic RG: application to condensed matter

- In the '90 rigorous RG was applied to condensed matter by Benfatto, Gallavotti (JSP 90) and Feldman, Trubowitz (HPA 90); interacting non relativistic fermions in the continuum. Extended singularity in $d \geq 2$; $D = 2 - n/2$, the theory is marginal and in $d \geq 2$ the rcc is function of the angles. In certain directions is marginally relevant (superconductivity). order by order perturbative analysis.

Rigorous fermionic RG: application to condensed matter

- In the '90 rigorous RG was applied to condensed matter by Benfatto, Gallavotti (JSP 90) and Feldman, Trubowitz (HPA 90); interacting non relativistic fermions in the continuum. Extended singularity in $d \geq 2$; $D = 2 - n/2$, the theory is marginal and in $d \geq 2$ the rcc is function of the angles. In certain directions is marginally relevant (superconductivity). order by order perturbative analysis.
- There was an attempt to construct non perturbatively the $T = 0$ properties in the continuum at $d = 2$ and proving superconductivity, using also sectors method (Feldman, Magnen, Rivasseau, Trubowitz EPL 1993); several interesting partial results but problem still (very!) open. Difficulty related to marginal relevance and infinitely many couplings.

Rigorous fermionic RG: application to condensed matter

- RG and sectors were used to construct the $T = 0$ properties in $d = 2$ of a special lattice model with asymmetric Fermi surface (asymmetry makes rcc effectively irrelevant) (Knoerrer Feldman Trubowitz CMP 2002), Fermi liquid behavior (regularity of counterterm proved order by order Feldman Trubowitz Salmhofer CPAA 1999)

Rigorous fermionic RG: application to condensed matter

- RG and sectors were used to construct the $T = 0$ properties in $d = 2$ of a special lattice model with asymmetric Fermi surface (asymmetry makes rcc effectively irrelevant) (Knoerrer Feldman Trubowitz CMP 2002), Fermi liquid behavior (regularity of counterterm proved order by order Feldman Trubowitz Salmhofer CPAA 1999)
- Temperature stops RG flow. RG and sectors used only to construct for T above exponentially small temperature Jellium continuum model (Disertori Rivasseau CMP 2000); then in the Hubbard model (Benfatto Giuliani Mastropietro AHP 2005) (temperature stops flow); proved regularity of counterterms.

Rigorous fermionic RG: application to condensed matter

- RG and sectors were used to construct the $T = 0$ properties in $d = 2$ of a special lattice model with asymmetric Fermi surface (asymmetry makes rcc effectively irrelevant) (Knoerrer Feldman Trubowitz CMP 2002), Fermi liquid behavior (regularity of counterterm proved order by order Feldman Trubowitz Salmhofer CPAA 1999)
- Temperature stops RG flow. RG and sectors used only to construct for T above exponentially small temperature Jellium continuum model (Disertori Rivasseau CMP 2000); then in the Hubbard model (Benfatto Giuliani Mastropietro AHP 2005) (temperature stops flow); proved regularity of counterterms.
- Going to lower temperatures is very hard; one can truncate the beta function and see numerically which kind of coupling increases more, see eg Honerkamp Salmhofer (PRB 2002), getting evidence for superconductivity for $d = 2$ attractive Hubbard (connected to the debated problem of high T_c supercondictivity)

Rigorous fermionic RG: application to condensed matter

- Actually the only cases in which the $T = 0$ properties of condensed matter model have been constructed are when the FS is pointlike; $d = 1$ (Luttinger liquids) and in $d \geq 2$ (graphene, weyl semimetals, Hall systems....); point like Fermi surface.

Rigorous fermionic RG: application to condensed matter

- Actually the only cases in which the $T = 0$ properties of condensed matter model have been constructed are when the FS is pointlike; $d = 1$ (Luttinger liquids) and in $d \geq 2$ (graphene, weyl semimetals, Hall systems....); point like Fermi surface.
- In the first case $D = 2 - n/2$ only 1 rcc; anomalous exponents, marginally marginal theory main difficulty vanishing of beta function.

- Actually the only cases in which the $T = 0$ properties of condensed matter model have been constructed are when the FS is pointlike; $d = 1$ (Luttinger liquids) and in $d \geq 2$ (graphene, weyl semimetals, Hall systems....); point like Fermi surface.
- In the first case $D = 2 - n/2$ only 1 rcc; anomalous exponents, marginally marginal theory main difficulty vanishing of beta function.
- In the second $D = 3 - n$ or $D = 4 - 3n/2$; coupling irrelevant main difficulty proving universality of transport coefficients.

Rigorous fermionic RG: Luttinger liquids, vanishing of beta function ad all that

- In $d = 1$ case the first proof of vanishing of beta function was given (Benfatto Gallavotti Mastropietro, PRB 1992, Benfatto Gallavotti Procacci Scoppola CMP 1994) using the exact solution of the Luttinger model by bosonization (Mattis Lieb 1966).

Rigorous fermionic RG: Luttinger liquids, vanishing of beta function ad all that

- In $d = 1$ case the first proof of vanishing of beta function was given (Benfatto Gallavotti Mastropietro, PRB 1992, Benfatto Gallavotti Procacci Scoppola CMP 1994) using the exact solution of the Luttinger model by bosonization (Mattis Lieb 1966).
- This was not (fully) satisfactory as would like to have a self-consistent proof avoiding "exact solutions" which are special (if even in $d = 1$ RG is not enough alone, no hope for future!).

Rigorous fermionic RG: Luttinger liquids, vanishing of beta function ad all that

- In $d = 1$ case the first proof of vanishing of beta function was given (Benfatto Gallavotti Mastropietro, PRB 1992, Benfatto Gallavotti Procacci Scoppola CMP 1994) using the exact solution of the Luttinger model by bosonization (Mattis Lieb 1966).
- This was not (fully) satisfactory as would like to have a self-consistent proof avoiding "exact solutions" which are special (if even in $d = 1$ RG is not enough alone, no hope for future!).
- Later Benfatto Mastropietro (CMP 2002, CMP 2004) developed a nechnique allowing to implement WI in each RG step; control of corrections due cut-off. This allowed the direct proof of vanishing of beta function. Subsequently a better choice of the "reference model" allowed to prove a version of the Adler-Bardeen theorem (Mastropietro JMP 2006) for chiral anomalies.

Rigorous fermionic RG: Luttinger liquids, vanishing of beta function ad all that

- Such methods allowed to rigously prove the Luttinger liquid relations proposed by Haldane (1980) in non solvable models (they were checked before only in solvable ones) (Benfatto Falco Mastropietro CMP 2008, PRL 2008); later this method was applied to interacting dimers (Giuliani Mastropietro Toninelli 2019) (also Pinson Spencer 2008) extending relations in the free case (Kenion Okunkov Sheffield Ann.Math 2002). Regularity+WI

Rigorous fermionic RG: Luttinger liquids, vanishing of beta function ad all that

- Such methods allowed to rigously prove the Luttinger liquid relations proposed by Haldane (1980) in non solvable models (they were checked before only in solvable ones) (Benfatto Falco Mastropietro CMP 2008, PRL 2008); later this method was applied to interacting dimers (Giuliani Mastropietro Toninelli 2019) (also Pinson Spencer 2008) extending relations in the free case (Kenion Okunkov Sheffield Ann.Math 2002). Regularity+WI
- The method also allowed to prove the axioms in Thirring and the Coleman equivalence (Benfatto Falco Mastropietro CMP 2007), extended recently (only in the free fermion point) (Bauerschmidt, Webb Eur. Math. Soc 2024)

Rigorous RG: Graphene, Hall insulators, quasi-periodic disorder

- In the case of graphene the FS is point like, and the interaction do not change the asymptotic behavior at $T = 0$ (Giuliani Mastropietro CMP 2008) .

Rigorous RG: Graphene, Hall insulators, quasi-periodic disorder

- In the case of graphene the FS is point like, and the interaction do not change the asymptotic behavior at $T = 0$ (Giuliani Mastropietro CMP 2008) .
- The main difficulty is to compute the trasnport coefficients; physical computaions (Mischenko 2009) showed a strong renormalization, in contrast with experiments. In Giuliani Mastropietro Porta CMP 2010 a theorem was proved establishing the exact universality. Cancellation of irrelevant terms.

Rigorous RG: Graphene, Hall insulators, quasi-periodic disorder

- In the case of graphene the FS is point like, and the interaction do not change the asymptotic behavior at $T = 0$ (Giuliani Mastropietro CMP 2008) .
- The main difficulty is to compute the trasnport coefficients; physical computaions (Mischenko 2009) showed a strong renormalization, in contrast with experiments. In Giuliani Mastropietro Porta CMP 2010 a theorem was proved establishing the exact universality. Cancellation of irrelevant terms.
- Hubbard interaction; for long range still open! (order by order lines of fixed points)

Rigorous RG: Graphene, Hall insulators, quasi-periodic disorder

- The universality of Hall conductance in presence of interaction was proved in Giuliani Porta Mastropietro CMP 2016 (see also Hasting Mikalakis CMP 2015 by topological methods).

Rigorous RG: Graphene, Hall insulators, quasi-periodic disorder

- The universality of Hall conductance in presence of interaction was proved in Giuliani Porta Mastropietro CMP 2016 (see also Hasting Mikalakis CMP 2015 by topological methods).
- Chiral Luttinger liquid behavior (see Wen, Froehlich,..) at the edges of Hall insulators and quantization in presence of interaction was proved in Antinucci Mastropetro Porta CMP 2022.

Rigorous RG: Graphene, Hall insulators, quasi-periodic disorder

- The universality of Hall conductance in presence of interaction was proved in Giuliani Porta Mastropietro CMP 2016 (see also Hasting Mikalakis CMP 2015 by topological methods).
- Chiral Luttinger liquid behavior (see Wen, Froehlich,..) at the edges of Hall insulators and quantization in presence of interaction was proved in Antinucci Mastropetro Porta CMP 2022.
- The presence of quasi periodic disorder (one particle Avila Jatomirskaja AoM 2008) inf 1d interacting fermions (Benfatto Gentile Mastropietro, JSP 1997 Mastropietro CMP 1998) was considered in the gapped case, using ideas from KAM Lindstedt series.

Rigorous RG: Graphene, Hall insulators, quasi-periodic disorder

- The universality of Hall conductance in presence of interaction was proved in Giuliani Porta Mastropietro CMP 2016 (see also Hasting Mikalakis CMP 2015 by topological methods).
- Chiral Luttinger liquid behavior (see Wen, Froehlich,..) at the edges of Hall insulators and quantization in presence of interaction was proved in Antinucci Mastropetro Porta CMP 2022.
- The presence of quasi periodic disorder (one particle Avila Jatomirskaja AoM 2008) inf 1d interacting fermions (Benfatto Gentile Mastropietro, JSP 1997 Mastropietro CMP 1998) was considered in the gapped case, using ideas from KAM Lindstedt series.
- In a gapless case (and no second Melnikov) in the 2D Ising model with quasi periodic disorder (Gallone Mastropietro CMP 2024) the Harris irrelevance conjectured by Luck JSP 1983 was proved.

Courses

- Proof of universality of the optical conductivity in graphene (Porta)

Courses

- Proof of universality of the optical conductivity in graphene (Porta)
- Proof of Harris-Luck criterion on quasi periodic Ising (Gallone)

Courses

- Proof of universality of the optical conductivity in graphene (Porta)
- Proof of Harris-Luck criterion on quasi periodic Ising (Gallone)
- Proof of Luttinger liquid construction (Mastropietro)

Courses

- Proof of universality of the optical conductivity in graphene (Porta)
- Proof of Harris-Luck criterion on quasi periodic Ising (Gallone)
- Proof of Luttinger liquid construction (Mastropietro)
- Conclusion and open problems

General introduction to the methods

- Benfatto,Gallavotti; Renormalization Group, Princeton Press (1996)
- Gentile, Mastropietro, Phys Rep 2001
- V. Mastropietro Non perturbative renormalization , World Scientific 2008

General introduction to the methods

- Benfatto,Gallavotti; Renormalization Group, Princeton Press (1996)
- Gentile, Mastropietro, Phys Rep 2001
- V. Mastropietro Non perturbative renormalization , World Scientific 2008
- See also for very related methods
- V. Rivasseau From perturbative to non perturbative renormalization, Princ Press 1991;
- M. Salmhofer Renormalization Springer 1999;
- R Bauerschmidt, DC Brydges, G Slade Introduction to a renormalisation group method Springer 2019;
- Giuliani, Mastropietro, Ryckov, Gentle introduction JHEP 2022